

A THEOREM ON FREE ENVELOPES

BY

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ABSTRACT. The free envelope of a finite commutative semigroup was defined by Grillet [Trans. Amer. Math. Soc. **149** (1970), 665–682] to be a finitely generated free commutative semigroup $F(S)$ with identity and a homomorphism $\alpha: S \rightarrow F(S)$ endowed with certain properties. Grillet raised the following question: does $\alpha(S)$ always generate a pure subgroup of the free Abelian group with the same basis as $F(S)$? We prove this is indeed the case. It follows as a result of two lemmas.

Lemma 1: Given a full rank proper subgroup H of a finitely generated free Abelian group F and a basis X of F there exists a surjective homomorphism $f: F \rightarrow \mathbb{Z}$ such that f is positive on X and $f|_H$ is not surjective. **Lemma 2:** A finitely generated totally cancellative reduced subsemigroup of a finitely generated free Abelian group F is contained in the positive cone of some basis of F . The following duality theorem is also proved. Let $S^* \cong \text{Hom}(S, N)$ where N is the nonnegative integers under addition. Then $S \cong S^{**}$ if and only if S is isomorphic to a unitary subsemigroup of a finitely generated free commutative semigroup with identity.

The free envelope of a finitely generated commutative semigroup S was introduced in [1]. It is a finitely generated free commutative semigroup $F(S)$ with identity together with a homomorphism $\alpha: S \rightarrow F(S)$ endowed with certain properties, in particular any homomorphism $f: S \rightarrow F$, where F is a finitely generated free commutative semigroup with identity, factors, not necessarily uniquely, through α .

The following question was raised in [1]: Does $\alpha(S)$ always generate a pure subgroup in the free Abelian group with the same basis as $F(S)$? We prove that this is indeed the case. As an application of this result, we prove the following duality theorem (where $S^* = \text{Hom}(S, N)$, and N is the nonnegative integers under addition): $S \cong S^{**}$ if and only if S is isomorphic to a unitary subsemigroup of a finitely generated free commutative semigroup with identity. The notation is the same as in [1], in particular, all semigroups are commutative, denoted additively, and identity elements are denoted by 0.

Recall that a commutative semigroup S is said to be totally cancellative if it is power cancellative (i.e., $na = nb$ implies $a = b$ for all $n \in N \setminus \{0\}$) and cancellative, and that a totally cancellative semigroup S is said to be reduced if either it has no identity element or its group of units is trivial.

The proof of the main theorem depends on the following lemmas.

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LEMMA 1. If F is a finitely generated free Abelian group, H is a subgroup of F of the same rank as F , and x_1, x_2, \dots, x_n is a free generating set of F , then there exists a surjective homomorphism $\beta: F \rightarrow Z$ such that:

- (1) $\beta|_H$ is not surjective;
- (2) $\beta(x_i) > 0$ for $i = 1, 2, \dots, n$.

PROOF. The proof will proceed by induction on the rank of F . If $\dim(F) = 1$, then $F \cong Z$. Thus define $\beta = \theta$ if $\theta(x_1) = 1$ and $\beta = -\theta$ otherwise, where θ is the isomorphism from F to Z above.

Assume the lemma for all free Abelian groups of rank $n - 1$, and let F have rank n . Since F/H is torsion there exists a smallest positive integer n_1 such that $n_1 x_i \in H$ for $i = 1, 2, \dots, n$.

Let F' be the free Abelian group generated by x_2, x_3, \dots, x_n , and let $H' = H \cap F'$.

Case (1). Assume $H' = F'$. Then $n_1 > 1$ since H is a proper subgroup of F . Let $\beta(x_1) = 1$ and $\beta(x_i) = n_1$ for $i = 2, 3, \dots, n$. Then $n_1 | \beta(h)$ for all $h \in H$. But β is surjective since $\beta(x_1) = 1$.

Case (2). Assume $H' \neq F'$.

Since $H' \subseteq F'$ contains $n_2 x_2, \dots, n_n x_n$, it has rank $n - 1$ and by the induction hypothesis there exists a surjective homomorphism $\beta': F' \rightarrow Z$ such that $\beta'|_{H'}$ is not surjective and $c_i = \beta'(x_i) > 0$ for $i = 2, 3, \dots, n$. Let c be the smallest positive integer in $\beta'(H')$. Then $\gcd_{i \geq 2}(c_i) = 1$ and $c > 1$. Clearly there exists a smallest k such that $0 < k < n_1$ and $(kx_1 + F') \cap H \neq \emptyset$. Thus there exists $\sum_{i \geq 2} p_i x_i$ such that $0 < p_i < n_i$ and $kx_1 + \sum_{i \geq 2} p_i x_i \in H$.

Let $j = \text{lcm}_{i \geq 2}(c_i)$ and $b = \gcd(k, \sum_{i \geq 2}(c_j n_i - c_i p_i))$. Define $c'_1 = (\sum_{i \geq 2} c_j n_i - c_i p_i)/b$ and $c'_i = (k/b)c_i$ for $i = 2, 3, \dots, n$. It is clear that $\gcd_{i \geq 1}(c'_i) = 1$ and $c'_i > 0$. Define $\beta(\sum_{i \geq 1} b_i x_i) = \sum_{i \geq 1} b_i c'_i$. Then β is surjective since $\gcd_{i \geq 1}(c'_i) = 1$ and $\beta(x_i) = c'_i$.

Let $x = \sum b_i x_i \in H$. If $b_1 = 0$ then $x \in H'$. Thus assume $b_1 \neq 0$, then $k|b_1$ by the choice of k , say $b_1 = sk$. Thus $x - s(kx_1 + \sum p_i x_i) \in H'$. Therefore, H is generated by H' and $kx_1 + \sum p_i x_i$. Hence

$$\langle H' \cup \{kx_1 + \sum p_i x_i\} \rangle = H.$$

By the definition of β it is clear that for all $r \in H'$, $c|\beta(r)$ and $\beta(kx_1 + \sum p_i x_i) = c(k/b)j \sum n_i$. Thus $c|\beta(kx_1 + \sum p_i x_i)$. Therefore, for all $r \in H$, $c|\beta(r)$, and thus the lemma is proved.

In what follows, $\langle x_1, \dots, x_n \rangle$ denotes the subsemigroup (not the subgroup) generated by x_1, \dots, x_n .

Let S and T be cancellative commutative semigroups, then $G(S)$ denotes the universal group of S , and if $f: S \rightarrow T$ is a homomorphism, then $G(f): G(S) \rightarrow G(T)$ is the induced homomorphism.

LEMMA 2. Let S be a finitely generated totally cancellative reduced semigroup which is embedded into a finitely generated free Abelian group F . There exists a basis of F , x_1, \dots, x_n , such that $S \subseteq \langle x_1, \dots, x_n \rangle$.

PROOF. From [1] there is a homomorphism $f: S \rightarrow \mathbb{N}$ which is positive on $S \setminus \{0\}$. Since S is both commutative and cancellative f can be extended to $G(f): G(S) \rightarrow \mathbb{Z}$. Let $G(S)^d$ be the divisible hull of $G(S)$. Then $G(f)$ can be extended to $f': G(S)^d \rightarrow \mathbb{Q}$. Since divisible groups are injective there is a homomorphism $g: G(F) \rightarrow \mathbb{Q}$ such that g is positive on $S \setminus \{0\}$. Let $T = \text{Ker}(g)$, the kernel of g . Then T has rank $n - 1$ since g has rank n , and hence $G(F)$ has a basis x_1, \dots, x_n such that $m_2 x_2, \dots, m_n x_n$ is a basis of T . Since $g(mx) = 0$ implies $g(x) = 0$, we see that x_2, \dots, x_n is a basis of T . Replacing x_1 by $-x_1$ if necessary we can also arrange that $g(x_1) > 0$. When $x = \sum p_i x_i \in S \setminus \{0\}$ we have $f(x) = p_1 f(x_1) > 0$. Thus $p_1 > 0$.

Let $\{s_1, s_2, \dots, s_k\}$ be a generating set of S , and for each i , $s_i = \sum_{j \geq 1} b_j^i x_j$. Then by the choice of x_1 , $b_1^i > 0$ for each i . Therefore, pick $p \in \mathbb{N}$ such that $b_j^i + p b_1^i > 0$ for all i and j . Define a new basis for F , x'_1, x'_2, \dots, x'_n , from the old basis x_1, x_2, \dots, x_n by $x'_1 = x_1 - \sum_{j \geq 2} p x_j$ and $x'_i = x_i$ for $i = 2, 3, \dots, n$. Thus

$$s_i = b_1^i x'_1 + \sum_{j \geq 2} (b_j^i + p b_1^i) x'_j,$$

and hence

$$S \subseteq \langle x'_1, x'_2, \dots, x'_n \rangle.$$

THEOREM 3. *If S is a finitely generated totally cancellative reduced semigroup, then $G(\alpha(S))$ is pure in $G(F(S))$, the universal group of the free envelope $(f(S), \alpha)$ of S .*

PROOF. Let U denote the subgroup of $G(F(S))$ generated by $\alpha(S)$. Let P denote the pure subgroup of $G(F(S))$ generated by $\alpha(S)$. Denote $P \cap F(S)$ by F . Now $U \subseteq P$, and let us assume that $P \neq U$. Then

(1) There exists a basis of P , x_1, x_2, \dots, x_n such that $\alpha(S) \subseteq \langle x_1, x_2, \dots, x_n \rangle$, by Lemma 2.

(2) There exists a surjective homomorphism $\beta: P \rightarrow \mathbb{Z}$ such that $\beta|_U$ is not surjective and $\beta(x_i) > 0$ for all i , by Lemma 1.

Let c be the smallest positive integer in the set $\beta(U)$. Then by (2), $c > 1$. Thus define $\beta'(a) = \beta(a)/c$ for all $a \in U$. Let $\beta'' = \beta'|_S$. Then by (1) and (2), β , β' and β'' are all positive on $S \setminus \{0\}$. Since β'' is the unique extension of β'' to U , and β' has a unique extension to $\gamma: P \rightarrow \mathbb{Q}$ (as $x \in P$ implies $nx \in U$ for some $n \in \mathbb{N}$). Since $\beta' = \beta/c$ on U , then $\gamma = \beta/c$, and because $c > 1$ and $\text{gcd}(\beta(P)) = 1$, γ must have nonintegral values. Therefore, β' cannot be extended to P (as an integral valued homomorphism). On the other hand, the quasi-universal property of free envelopes says that β'' and hence β' can be extended to $F(S)$ and thus to $G(F(S)) \supseteq P$, a contradiction. Therefore, $U = P$, and hence U is pure in $G(F(S))$ which completes the proof of the theorem.

A subsemigroup S of a semigroup K is said to be unitary if $x + a \in S$ and $a \in S$ implies $x \in S$ for all $x, a \in S$.

DEFINITION. If S is a subsemigroup of a semigroup K , then the *unitary closure* of S with respect to K , $U_K(S)$, is the smallest unitary subsemigroup of K containing S . Note: If S is a finitely generated totally cancellative reduced semigroup which is contained in a finitely generated free semigroup F , then $U_K(S) = G(S) \cap F$.

DEFINITION. Let S be a subsemigroup of a cancellative semigroup K , and let P be the pure subgroup of $G(K)$ generated by S . Then the *pure unitary closure* of S in K , $P_K(S)$, is $P \cap K$.

COROLLARY 4. If S is a finitely generated totally cancellative reduced semigroup, then the unitary closure of $\alpha(S)$ in the free envelope $(F(S), \alpha)$ of S coincides with the pure unitary closure of $\alpha(S)$ in $F(S)$.

PROOF. This follows directly from the theorem.

DEFINITION. Let S be a semigroup, and N be the positive integers under addition. Then the *dual* of S , S^* , is equal to $\text{Hom}(S, N^0)$. From [1] it is known that:

- (1) $S^* \cong S^{***}$;
- (2) The evaluation map $\sigma: S \rightarrow S^{**}$ is injective if S is finitely generated totally cancellative reduced;
- (3) If $f: S^{**} \rightarrow T$ is injective when restricted to $\sigma(S)$, then f is injective;
- (4) If $(F(S), \alpha)$ is the free envelope of S , then there exists an isomorphism, K , such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S^{**} \\ & \searrow \alpha & \nearrow K \\ & P_{F(S)}(\alpha(S)) & \end{array}$$

- (5) If $f: S^{**} \rightarrow T$ is a homomorphism and $f|_{\sigma(S)}$ is an epimorphism, then f is an epimorphism.

COROLLARY 5. If S is a finitely generated totally cancellative reduced semigroup, then $S^{**} \cong U_{F(S)}(\alpha(S))$.

PROOF. The proof follows directly from Corollary 4 and Proposition 3.7 of [1].

COROLLARY 6. Let S be a finitely generated totally cancellative reduced semigroup. Then $S \cong S^{**}$. If and only if S can be embedded as a unitary subsemigroup of a finitely generated free semigroup.

PROOF. Let $S \cong S^{**}$. Then by Corollary 4, $S \cong S^{**} U_{F(S)}(\alpha(S))$, and $U_{F(S)}(\alpha(S))$ is unitary in $F(S)$.

Conversely, assume that S is a unitary subsemigroup of a finitely generated free semigroup F . Then by hypothesis $S = \bar{S}_F$. Therefore, there exists a homomorphism $f: F(S) \rightarrow F$ such that the following diagram commutes:

$$\begin{array}{ccc}
 U_{F(S)}(\alpha(S)) & \subseteq & F(S) \\
 \uparrow \alpha & & \downarrow f \\
 S & \subseteq & F \\
 & \downarrow j &
 \end{array}$$

Hence $f\alpha = 1_S$.

If $x \in F(S)$, $\alpha(s) \in \alpha(S)$ and $x + \alpha(s) \in \alpha(S)$, then $x \in U_{F(S)}(\alpha(S))$. Hence $a + x \in \alpha(S)$ for some $a \in \alpha(S)$. Thus $f(a) + f(x) \in S$. Therefore, $f(s) \in S$. Thus $U_{F(S)}(\alpha(S)) \subseteq S$. Thus there is an induced homomorphism $g: U_{F(S)}(\alpha(S)) \rightarrow S$ induced by the following commutative diagram:

$$\begin{array}{ccccc}
 & & U_{F(S)}(\alpha(S)) & \subseteq & F(S) \\
 & \nearrow \alpha & & \downarrow i & \downarrow f \\
 S & & & & \\
 & \searrow g & & \downarrow j & \\
 & & S & \subseteq & F
 \end{array}$$

Therefore, $g\alpha = 1_S$ and also $\alpha g\alpha = \alpha = 1_{U_{F(S)}(\alpha(S))}\alpha$. Thus $\alpha g = 1_{U_{F(S)}(\alpha(S))}$, by property (5) above. Therefore, by Corollary 4, $S \cong U_{F(S)}(\alpha(S)) \cong S^{**}$.

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